JGP - Vol. 1, n. 2, 1984

# A geometrical version of the higher order Hamilton formalism in fibred manifolds

IVAN KOLÁŘ
Institute of Mathematics of the ČSAV
BRNO-CZECHOSLOVKIA

**Abstract.** *It has been clarified recently that an $r$-th order Lagrangian on a fibred manifold $Y \to X$ does not determine a unique Poincaré-Cartan form provided* $\dim X > 1$ *and* $r > 2$, [1], [4], [6], [9], [10]. *To make this fact more transparent, we introduced a new operation generalizing the formal exterior differentiation,* [6]. *In the present paper we deduce in such a way that a unique Poincaré-Cartan form can be determined by means of a simple additional structure - a linear symmetric connection $\Gamma$ on the base manifold $X$ (or, more generally, by a convenient splitting $S$). Then we present a suitable geometric definition of a regular $r$-th order Lagrangian on $Y$ and we prove that any our Poincaré-Cartan form can be used in a geometrical version of the higher order Hamilton formalism.*

The starting point for this paper was a lecture presented in the Banach Center at Warsaw during the 1983 Trimester on Mathematical Physics. The author is grateful to Prof. A.Trautman for his interest in the lecture and his comments.

## 1. DECOMPOSITION FORMULA

All manifolds and maps are assumed to be infinitely differentiable and all morphisms of fibred manifolds are base-preserving. – Given a fibred manifold $\pi : Y \to X$, we denote by $\pi_r : J^r Y \to X$ its $r$-th jet prolongation and by $\pi^s_r : J^r Y \to J^s Y$, $0 \leqslant s \leqslant r$, $(J^0 Y = Y)$ the jet projections. If $x^i$, $y^p$, $i, j, \ldots = 1,$

---

This article is based on lectures given by the Author during the Trimester on Mathematical Physics at the Stefan Banach International Mathematical Centre, Warsaw, Sept. - Nov. 1983

$\ldots, n = \dim X$, $p, q = 1, \ldots, m = \dim Y - \dim X$, are some local fibre coordinates on $Y$, then the induced coordinates on $J^r Y$ are $x^i, y^p_\alpha$ for all multiindices $|\alpha| \leqslant r$. As usual, $\alpha + \beta$ means the sum of two multiindices, $(\alpha + \beta)_i = \alpha_i + \beta_i$. Any ordinary index $i$ can be interpreted as a multiindex with $i$-th component 1 and all other components 0. Since we have to discuss some problems of tensorial character, we shall also use the classical notation of the tensor calculus. In such a situation we write $y^p_\alpha = y^p_{j_1 \ldots j_k}$ for $\alpha = j_1 + \ldots + j_k$. We use the summation convention for ordinary indices, but we always indicate explicitely the summation with respect to multiindices. We set $\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$, $\omega_i = \dfrac{\partial}{\partial x^i} \lrcorner\, \omega$.

For every morphism $\varphi : J^r Y \to \wedge^k T^*X$, one defines its formal exterior differential $D\varphi : J^{r+1} Y \to \wedge^{k+1} T^*X$ by $(j^{r+1}s)^* D\varphi = \mathrm{d}((j^r s)^* \varphi)$ for every local section $s$ of $Y$, [13]. If the local coordinate expression of $\varphi$ is $\varphi = a_{i_1 \ldots i_k}(x^i, y^p_\alpha)\, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$, then $D\varphi = D_j a_{i_1 \ldots i_k} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$, where $D_j f = \partial_j f + \sum_{|\alpha| \leqslant r} (\partial^\alpha_p f) y^p_{\alpha + j}$, $\partial_j = \partial/\partial x^j$, $\partial^\alpha_p = \partial/\partial y^p_\alpha$, denotes the formal (or total) derivative of a function $f : J^r Y \to \mathbf{R}$. Clearly, $DD\varphi = 0$. Any vertical vector field $\eta$ on $Y$ induces a vector field $J^r\eta$ on $J^r Y$ such that $\exp(tJ^r\eta) = J^r(\exp t\eta)$, where $\exp t\xi$ means the flow of a vector field $\xi$. In coordinates, if $\eta = \eta^p(x, y)\, \partial_p$, then $J^r \eta = \sum_{|\alpha| \leqslant r} (D_\alpha \eta^p)\, \partial^\alpha_p$. This implies directly: for every morphism $A : J^r Y \to V^* J^s Y \otimes \wedge^k T^*X$ over the identity of $J^s Y$, $s \leqslant r$, there exists a unique morphism $\mathscr{D}A : J^{r+1} Y \to V^* J^{s+1} Y \otimes \wedge^{k+1} T^*X$ satisfying

$$\langle \mathscr{D}A, J^{s+1}\eta \rangle = D \langle A, J^s \eta \rangle \tag{1}$$

for every vertical vector field $\eta$ on $Y$, [6]. Obviously, $\mathscr{D}\mathscr{D}A = 0$. For $k = n - 1$, we write $A = \sum_{|\alpha| \leqslant s} a^{\alpha i}_p\, \mathrm{d}y^p_\alpha \otimes \omega_i$, where $\alpha i$ is a pair of a multiindex and an ordinary index, and we have

$$\mathscr{D}A = \sum_{|\alpha| \leqslant s} [(D_i a^{\alpha i}_p)\, \mathrm{d}y^p_\alpha + a^{\alpha i}_p\, \mathrm{d}y^p_{\alpha + i}] \otimes \omega. \tag{2}$$

We define an $r$-th order Lagrangian on $Y$ as a morphism $\lambda : J^r Y \to \wedge^n T^*X$, $\lambda = L(x^i, y^p_\alpha)\omega$, [13]. Its vertical differential $\delta\lambda = \sum_{|\alpha| \leqslant r} (\partial^\alpha_p L)\, \mathrm{d}y^p_\alpha \otimes \omega$ can be interpreted as a map $J^r Y \to V^* J^r Y \otimes \wedge^n T^*X$. For the tensorial considerations we introduce

$$L^{j_1 \ldots j_k}_p = \frac{\alpha!}{k!}\, \partial^\alpha_p L \qquad \text{for} \qquad \alpha = j_1 + \ldots + j_k. \tag{3}$$

Then we have

(4) $$\delta\lambda = (L_p \mathrm{d}y^p + \ldots + L_p^{j_1 \cdots j_r} \mathrm{d}y^p_{j_1 \cdots j_r}) \otimes \omega.$$

We clarified in [6] and [9] that a basic problem is to discuss a decomposition

(5) $$\delta\lambda = \mathscr{D}M + E$$

with $M : J^{2r-1}Y \to V^*J^{r-1}Y \otimes \Lambda^{n-1}T^*X$ and $E : J^{2r}Y \to V^*Y \otimes \Lambda^n T^*X$ (the injection $V^*Y \to V^*J^rY$ being tacitly used here).

Write $M = \sum_{|\alpha| \leqslant r-1} b_p^{\alpha i} \mathrm{d}y^p_\alpha \otimes \omega_i$. Similarly to (3), we set $B_p^{j_1 \cdots j_k i} = \frac{\alpha!}{k!} b_p^{\alpha i}$, $\alpha = j_1 + \ldots + j_k$, so that $B$'s are symmetric in $j_1, \ldots, j_k$, not in $i$. Then

(6) $$M = (B_p^i \mathrm{d}y^p + \ldots + B_p^{j_1 \cdots j_{r-1} i} \mathrm{d}y^p_{j_1 \cdots j_{r-1}}) \otimes \omega_i.$$

Decomposition (5) leads to the following equations

(7) $$\begin{aligned} L_p^{j_1 \cdots j_r} &= B_p^{(j_1 \cdots j_r)} \\ &\vdots \\ L_p^{j_1 \cdots j_k} &= D_i B_p^{j_1 \cdots j_k i} + B_p^{(j_1 \cdots j_k)} \\ &\vdots \\ L_p &= D_i B_p^i + \mathrm{e}_p \end{aligned}$$

with $E = \mathrm{e}_p \, \mathrm{d}y^p \otimes \omega$.

Evaluating $\mathrm{e}_p$ by a backward procedure, we find for any $B$'s

(8) $$\mathrm{e}_p = \sum_{|\alpha| \leqslant r} (-1)^{|\alpha|} D_\alpha \partial_p^\alpha L,$$

so that the Euler morphism $E$ is uniquely determined. On the other hand, $M$ cannot be unique in general: if we take any $C : J^{2r-2}Y \to V^*J^{r-2}Y \otimes \Lambda^{n-2}T^*X$, then $M + \mathscr{D}C$ also satisfies (5) by virtue of $\mathscr{D}\mathscr{D}C = 0$. In [9] we proved the global existence of such a morphism $M$ and we also deduced a converse assertion: if $M$ and $\overline{M}$ are two morphisms satisfying (5), then there exists a morphism $C : J^{2r-2}Y \to V^*J^{r-2}Y \otimes \Lambda^{n-2}T^*X$ such that $\overline{M} = M + \mathscr{D}C$. In particular, $M$ is unique for $r = 1$ and any $n$ or $n = 1$ and any $r$.

Define a vector bundle $K^s_{r-1}$ over $J^{r-1}Y$ by an exact sequence

(9) $$0 \to K^s_{r-1} \to VJ^{r-1}Y \xrightarrow{V\pi^s_{r-1}} (\pi^s_{r-1})^! V^*J^sY \to 0$$

where $(\pi^s_{r-1})^!$ denotes the pullback over $J^{r-1}Y$. For $s = r-2$, the fibers of $K^{r-2}_{r-1}$ are $VY \otimes S^{r-1}T^*X$. If we compose the dual map $V^*J^{r-1}Y \to K^{r-2*}_{r-1}$ with $M$, we obtain $\widetilde{M} : J^{2r-1}Y \to V^*Y \otimes S^{r-1}TX \otimes \Lambda^{n-1}T^*X$ and we can require $(\mathrm{id} \otimes \lrcorner) \circ \widetilde{M} : J^{2r-1}Y \to V^*Y \otimes S^{r-2}TX \otimes \Lambda^{n-2}T^*X$ to vanish. The coordinate meaning of this condition is $B_p^{j_1 \dots j_{r-2} ji} = B_p^{j_1 \dots j_{r-2} ij}$ and such a $M$ will be said to be quasisymmetric. For $r = 2$, (7) implies that there is a unique quasisymmetric $M$ satisfying $\delta\lambda = \mathscr{S}M + E$. Its coordinate expression is

$$[(L^i_p - D_j L^{ji}_p)\mathrm{d}y^p + L^{ji}_p \mathrm{d}y^p_j] \otimes \omega_i . \tag{10}$$

However, one cannot continue in such a procedure. For $r = 3$ we deduced by direct evaluation that the condition $B^{ij}_p = B^{ji}_p$ depends on the coordinate system (an obstruction being formed by the second partial derivatives of the transformation on the base manifold), [10], p. 207, [6], p. 473.

Hence a natural problem is how to determine a unique $M$ by means of an additional structure. We shall show that it sufficies to add a linear symmetric connection $\Gamma$ on $X$, which is more economical than the pairs of connections used in [4] and [1]. To clarify the basic idea of our construction, we first consider any splitting $S : T^*X \to T^{r-1*}X$, where $T^{r-1*}X = J^{r-1}(X, \mathbf{R})_0$, which is a vector bundle over $X$. In other words, $S$ is a linear morphism such that $p \circ S = \mathrm{id}$, where $p : T^{r-1*}X \to T^*X$ is the canonical projection. Then we shall show that every $\Gamma$ determines a splitting $\Gamma_{r-1} : T^*X \to T^{r-1*}X$.

Given any vector bundle $E \to X$, we define $E^0_s$ by an exact sequence

$$0 \to E^0_s \to J^s E \to E \to 0. \tag{11}$$

A canonical map $\varkappa : J^{s-1}E \otimes T^{s*}X \to E^0_s$ can be constructed as follows, [11], [7]. Having $H = j^{s-1}_x \sigma \in J^{s-1}E$ and $F = j^s_x f \in T^{s*}X$, $f\sigma$ is a section of $E$. Obviously, $f(x) = 0$ implies that $j^s_x(f\sigma)$ depends on $H$ and $F$ only and $j^s_x(f\sigma) \in E^0_s$. This gives a bilinear map from the Whitney sum $J^{s-1}E \oplus T^{s*}X$ into $E^0_s$ inducing $\varkappa$. In coordinates, if $H = \sum_{|\beta| \leq s-1} \frac{1}{\beta!} h^p_\beta x^\beta$, $F = \sum_{0 < |\gamma| \leq s} \frac{1}{\gamma!} a_\gamma x^\gamma$ and we set $\varkappa(H \otimes F) = \sum_{0 < |\alpha| \leq s} \frac{1}{\alpha!} g^p_\alpha x^\alpha$, then

$$g^p_\alpha = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta!\gamma!} h^p_\beta a_\gamma \tag{12}$$

where the sum is taken over all multiindicial decompositions of $\alpha$. If we add a splitting $S : T^*X \to T^{s*}X$, $a_\gamma = S^i_\gamma a_i$, $S^i_j = \delta^i_j$, we obtain a map $\varkappa \circ (\mathrm{id} \otimes S) : J^{s-1}E \otimes T^*X \to E^0_s$ with the following coordinate expression

$$g^p_\alpha = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta!\gamma!} S^i_\gamma h^p_\beta a_i . \tag{13}$$

Given a fibred manifold $Y$, one analogously introduces a map

(14) $$\varkappa : (\pi^{s}_{s-1})^{!}\, VJ^{s-1}Y \otimes T^{s*}X \to VJ^{s}Y$$

by means of the canonical identification $VJ^sY \approx J^sVY$, [5]. Every $G \in VJ^sY$ corresponds to an $s$-jet $j^s_x\sigma$, where $\sigma$ is a local section of $VY \to X$. Then $j^s_x(f\sigma)$ is identified with an element of $K^0_s$. This induces $\varkappa$ by bilinearity. If we add a splitting $S : T^*X \to T^{s*}X$, then (13) implies that the coordinate expression of $\tilde{S} = \varkappa \circ (\mathrm{id} \otimes S) : (\pi^{s-1}_{s})^{!}\, VJ^{s-1}Y \otimes T^*X \to VJ^sY$ is

(15) $$\mathrm{d}y^p_\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\, S^i_\gamma\, \mathrm{d}y^p_\beta \otimes \partial_i$$

Every linear symmetric connection $\Gamma$ on $X$ determines a splitting $\Gamma_s : T^*X \to T^{s*}X$ as follows. Every $a \in T^*_xX$ can be interpreted as a linear map $\lambda(a) : T_xX \to \mathbf{R}$. Consider the exponential map exp of $\Gamma$ restricted to a diffeomorphism from a neighbourhood $0 \in U \subset T_xM$ into a neighbourhood $x \in V \subset X$. Then $\lambda(a) \circ \exp^{-1} : V \to \mathbf{R}$ and we define $\Gamma_s(a) = j^s_x(\lambda(a) \circ \exp^{-1})$.

Given a splitting $S : T^*X \to T^{r-1*}X$, a morphism $M : J^{2r-1}Y \to V^*J^{r-1}Y \otimes \Lambda^{n-1}T^*X$ will be called $S$-quasisymmetric, if the resulting map in the following diagram vanishes

$$\begin{array}{ccc}
J^{2r-1}Y & \xrightarrow{\ M\ } & V^*J^{r-1}Y \otimes \Lambda^{n-1}T^*X \\
 & \searrow & \downarrow S^* \otimes \mathrm{id} \\
 & & (\pi^{r-2}_{r-1})^{!}\, V^*J^{r-2}Y \otimes TX \otimes \Lambda^{n-1}T^*X \\
 & & \downarrow \mathrm{id} \otimes \lrcorner \\
 & & (\pi^{r-2}_{r-1})^{!}\, V^*J^{r-2}Y \otimes \Lambda^{n-2}T^*X.
\end{array}$$

In particular, if $\Gamma$ is an integrable connection and $x^i$ is an affine local coordinate system of $\Gamma$, then $\Gamma_{r-1}$ is determined by $S^i_\beta = 0$, $|\beta| \geqslant 2$. Then one finds easily that $M$ is $\Gamma_{r-1}$-quasisymmetric if and only if

(17) $$B^{j_1 \cdots j_s ji}_p = B^{j_1 \cdots j_s ij}_p \qquad \text{for all } s.$$

This is a justification of our terminology.

PROPOSITION 1. *For every splitting $S : T^*X \to T^{r-1*}X$ there exists a unique $S$-quasisymmetric morphism $M_S : J^{2r-1}Y \to V^*J^{r-1}Y \otimes \Lambda^{n-1}T\ X$ satisfying $\delta\lambda = \mathscr{D}M_S + E$.*

*Proof.* Take a local coordinate system $x^i$ on $X$ and consider first the local splitting determined by the integrable connection corresponding to $x^i$. Then the symmetries (17) imply that all $B$'s in (7) are uniquely determined and

$$B_p^{j_1 \dots j_k i} = L_p^{j_1 \dots j_k i} - D_s L_p^{s j_1 \dots j_k i} + \dots \tag{18}$$
$$+ (-1)^s D_{s_1 \dots s_s} L_p^{s_1 \dots s_s j_1 \dots j_k i}$$

with $s + k + 1 = r$. For an arbitrary splitting $S$, we deduce by (15) $B_p^{j_1 \dots j_k ji} = B_p^{j_1 \dots j_k ij}$ + certain linear combinations of the products of $B$'s with more superscripts with some $S$'s. Then (7) gives

$$B_p^{j_1 \dots j_k} = L_p^{j_1 \dots j_k} - D_i B_p^{j_1 \dots j_k i} + C_p^{j_1 \dots j_k} \tag{19}$$

where $C_p^{j_1 \dots j_k}$ is a certain linear combination with rational coefficients of the products $B_p^{i_1 \dots i_k \; 2 s_1 \dots s_s i} S^j_{s_1 \dots s_s}$ (no summation), $s \geqslant 2$. This determines $M_S$.
Q.E.D.

The morphism $M_S$ will be called the Poincaré-Cartan morphism of $\lambda$ determined by $S$. For $S = \Gamma_{r-1}$ we say that $M_{\Gamma_{r-1}} = M_\Gamma$ is determined by $\Gamma$. Since the global existence of a linear symmetric connection on every $X$ is a well-known fact, Proposition 1 gives another proof of the global existence of the Poincaré-Cartan morphisms.

For $r = 2$ one can take the identity map of $T^*X$ only, which gives (10). For $r = 3$, any splitting $T^*X \to T^{2*}X$ coincides with a linear symmetric connection $\Gamma$ on $X$. In general, local coordinates $x^i$ or $X$ induce the additional coordinates $f_{i_1 \dots i_k} = \partial_{i_1} \dots \partial_{i_k} f$ on $T^{s*}X$, $k = 1, \dots, s$. If $f_{ij} = \Gamma^k_{ij} f_k$ are the equations of $\Gamma$, then the coefficients of $M_\Gamma$ are

$$B_p^{jks} = L_p^{jks}, \; B_p^{ji} = L_p^{ji} - D_k L_p^{jik} + L_p^{ks(j}\Gamma_{ks}^{i)}, \tag{20}$$
$$B_p^i = L_p^i - D_j L_p^{ji} + D_{jk} L_p^{jki} - D_j L_p^{ks(j}\Gamma_{ks}^{i)}.$$

We now describe an algorithm for finding the equations of the splitting $\Gamma_s$: $T^*X \to T^{s*}X$ determined by $\Gamma$ in an arbitrary local coordinate system on $X$ (in the normal coordinate system of $\Gamma$ at $x \in X$, the equations of $\Gamma_s$ at $x$ are $f_{ij} = 0, \dots, f_{i_1 \dots i_s} = 0$ by definition). This is based on the fact that the second and higher order derivatives of $\lambda(a) \circ \exp^{-1}$ along each geodesics passing through $x$ vanish. We find the explicit formulae for $\Gamma_2$ and $\Gamma_3$. The equations of geodesics are

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \tag{21}$$

and the second derivative of a function $f$ along a curve $x^i(t)$ is

(22) $$\frac{\partial^2 f}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{\partial f}{\partial x^i} \frac{d^2 x^i}{dt^2}.$$

Hence $f_{ij} = \Gamma^k_{ij} f_k$. In the third order we have to evaluate $\frac{d^3 x^i}{dt^3}$ from (21) and differentiate once more (22). Then our condition gives

(23) $$f_{ijk} = [\partial_{(k} \Gamma^{\varrho}_{ij)} + \Gamma^{\varrho}_{m(i} \Gamma^m_{jk)}] f_{\varrho}.$$

In this way, one can proceed step by step on.

## 2. REGULAR LAGRANGIANS

Our definition of a regular $r$-th order Lagrangian is motivated by Proposition 2 below and our requirements are stronger than in [2]. Our approach is inspired by Shadwick, [12], but we use a direct geometric construction applied to the Lagrangian itself (we remark that the results mentioned in §1 imply that Shadwick's momenta are not geometrically well defined for $r > 2$ and $n > 1$).

If we restrict $\delta\lambda$ to $K^{r-1}_r$, we obtain a linear map $\Lambda(u) : VY \otimes S^r T^*X \to \bigwedge^n T^*X$, or an element $\Lambda(u) \in V^*Y \otimes S^r TX \otimes \bigwedge^n T^*X$, for every $u \in J^r Y$. The pullback $L_r Y = (\pi^0_{r-1})^! \; V^*Y \otimes S^r TX \otimes \bigwedge^n T^*X$ over $J^{r-1}Y$ will be called the $r$-th Legendre bundle of $Y$ and $\Lambda : J^r Y \to L_r Y$ will be called the Legendre transformation of $\lambda$ (the first order case was studied in [8]). If we introduce fibre coordinates on $L_r Y$ by the decomposition $s^{j_1 \dots j_r}_p \, dy^p \otimes \left( \frac{\partial}{\partial x^{j_1}} \circ \dots \circ \frac{\partial}{\partial x^{j_r}} \right) \otimes \omega$, then the coordinate expression of the Legendre transformation is $x^i = x^i$, $y^p_\alpha = y^p_\alpha$, $|\alpha| \leqslant r-1$ and $s^{j_1 \dots j_r}_p = L^{j_1 \dots j_r}_p$. The vertical differential of $\Lambda$ (over $J^{r-1}Y$) $\delta\Lambda : K^{r-1}_r \to L_r Y$ can be viewed as a map $\delta\Lambda : J^r Y \to V^*Y \otimes S^r TX \otimes V^*Y \otimes S^r TX \otimes \bigwedge^n T^*X$. For every $1 \leqslant k \leqslant r$, consider the canonical map

(24) $$s_k : S^r TX \otimes S^r TX \to (S^{2r-k} TX) \otimes S^k TX.$$

Then the induced map $\delta_k \Lambda = (\mathrm{id} \otimes s_k) o \delta\Lambda$ can be interpreted as

(25) $$\delta_k \Lambda : J^r Y \to \mathrm{Hom}\,(VY \otimes S^k T^*X \otimes \bigwedge^n TX, \; V^*Y \otimes S^{2r-k} TX).$$

DEFINITION. *A Lagrangian $\lambda$ will be called $k$-regular, if the linear map $(\delta_k \Lambda)(u) : VY \otimes S^k T^*X \otimes \bigwedge^n TX \to V^*Y \otimes S^{2r-k} TX$ is a monomorphism for every $u \in J^r Y$. A Lagrangian is said to be regular if it is $k$-regular for every $1 \leqslant k \leqslant r$.*

For any $\alpha = i_1 + \dots + i_r$, we set

$$(26)\qquad L_{qp}^{j_1\dots i_r j_1\dots j_r} = \frac{\alpha!}{r!}\,\partial_q^\alpha L_p^{j_1\dots j_r}.$$

Since the coordinate form of $s_k$ is $x_{i_1\dots i_r j_1\dots j_r} \to x_{(i_1\dots i_r j_1\dots j_k) j_{k+1}\dots j_r}$, the $k$-regularity of $\lambda$ is equivalent to the following condition. If

$$(27)\qquad L_{qp}^{i_1\dots i_r j_1\dots j_r} z^q_{i_1\dots i_r j_1\dots j_k} w^p_{j_{k+1}\dots j_r} = 0$$

for every $z^q_{i_1\dots i_r j_1\dots j_k} \in V^*Y \otimes S^{2r-k}TX$, then $w^p_{j_{k+1}\dots j_r} = 0$. Clearly, $\lambda$ is $r$-regular if and only if $\Lambda$ is a local diffeomorphism.

Let $\psi : TJ^rY \to VJ^{r-1}Y$ be the structure form of $J^rY$, $\psi = \sum_{|\alpha| \leqslant r-1} \omega^p_\alpha \partial^\alpha_p$, $\omega^p_\alpha = \mathrm{d}y^p_\alpha - y^p_{\alpha+i}\,\mathrm{d}x^i$, [3], [5]. For any splitting $S$, we define an exterior $n$-form $\psi \barwedge M_S$ on $J^{2r-1}Y$ by natural combination of the contraction with respect to $VJ^{r-1}Y$ and alternation, $\psi \barwedge M_S = \sum_{|\alpha| \leqslant r-1} b^{\alpha i}_p \omega^p_\alpha \wedge \omega_i$, [6]. If we interpret $\lambda$ as an exterior $n$-form on $J^rY$, then $\theta_S = \lambda + \psi \barwedge M_S$ is an exterior $n$-form on $J^{2r-1}Y$, which will be called the Poincaré-Cartan form of $\lambda$ determined by $S$ (or by $\Gamma$ in the case $S = \Gamma_{r-1}$).

In general, a section $u : X \to J^sY$ is said to be $k$-holonomic, $k \leqslant s$, if $\pi^k_s u = j^k(\pi^0_s u)$.

PROPOSITION 2. *Let $\lambda$ be a regular Lagrangian and $S$ any splitting. If a section $u : X \to J^{2r-1}Y$ satisfies*

$$(28)\qquad u^*(\zeta \lrcorner \mathrm{d}\theta_S) = 0$$

*for every $\pi^r_{2r-1}$-vertical vector field $\zeta$ on $J^{2r-1}Y$, then $u$ is $r$-holonomic.*

*Proof.* By (18) and (19), every $B_p^{j_1\dots j_k i}$ is a affine map from the affine bundle $J^{r+s}Y \to J^{r+s-1}Y$ into reals, $k + 1 + s = r$, the linear part of which is

$$(29)\qquad (-1)^s L_{qp}^{i_1\dots i_r s_1\dots s_s j_1\dots j_k i} y^q_{i_1\dots i_r s_1\dots s_s}$$

independently on $S$. (In other words, (29) is the highest order part of $B_p^{j_1\dots j_k i}$). Let $u^p_{i_1\dots i_k}$ be the coordinate functions of $u$. Set $u^*(\omega^p_{j_1\dots j_k} \wedge \omega_i) = U^p_{j_1\dots j_k i}\omega$, so that $U^p_{j_1\dots j_k i} = \partial_i u^p_{j_1\dots j_k} - u^p_{j_1\dots j_k i}$. Consider first a $\pi^{2r-2}_{2r-1}$-vertical vector field $\zeta_1$. Then the equation $u^*(\zeta_1 \lrcorner \mathrm{d}\theta_S) = 0$ reads

$$(30)\qquad L_{qp}^{i_1\dots i_r s_1\dots s_{r-1} i} \zeta^q_{i_1\dots i_r s_1\dots s_{r-1}} U^p_i = 0.$$

Since $\lambda$ is 1-regular, (30) implies $U^p_i = 0$, so that $u$ is 1-holonomic. Assume by induction that $u^*(\zeta_k \lrcorner \mathrm{d}\theta_S) = 0$ for every $\pi^{2r-k-1}_{2r-1}$-vertical vector field $\zeta_k$ implies that $u$ is $k$-holonomic, i.e. $u^*\omega^p_\alpha = 0$ for all $|\alpha| \leqslant k-1$. By (29), the condition $u^*(\zeta_{k+1} \lrcorner \mathrm{d}\theta_S) = 0$ for a $\pi^{2r-k}_{2r-1}$-vertical vector field $\zeta_{k+1} =$

$\sum_{r+s \leqslant |\alpha| \leqslant 2r-1} \zeta^p_\alpha \partial^\alpha_p$ reads

(31) $$L^{i_1 \dots i_r \ell_1 \dots \ell_s j_1 \dots j_k i}_{qp} \zeta^q_{i_1 \dots i_r \ell_1 \dots \ell_s} U^p_{j_1 \dots j_k i} = 0.$$

Since $\lambda$ is $(k+1)$-regular, (31) implies $U^p_{(j_1 \dots j_k i)} = 0$. We have $u^p_{j_1 \dots j_k} = \partial_{j_1} \dots \partial_{j_k} u^p$ by the induction hypothesis. Then $U^p_{(j_1 \dots j_k i)} = 0$ implies $u^p_{j_1 \dots j_k i} = \partial_i \partial_{j_1} \dots \partial_{j_k} u^p$. Hence $u$ is $(k+1)$-holonomic. In the last step of this procedure we obtain that $u$ is $r$-holonomic. Q.E.D.

REMARK 1. For $n = 1$ a stronger result holds: if $\lambda$ is a regular Lagrangian with the (unique) Poincaré-Cartan form $\theta$ and $u^*(\zeta \lrcorner \mathrm{d}\theta) = 0$ for every $\pi^0_{2r-1}$-vertical vector field $\zeta$ on $J^{2r-1}Y$, then $u$ is $(2r-1)$-holonomic, [6].

## 3. A GEOMETRICAL FORM OF THE HIGHER ORDER HAMILTON FORMALISM

A section $s : X \to Y$ is said to be a critical section of $\lambda$ if $(j^{2r}s)^*E = 0$. (In coordinates, $(j^{2r}s)^*\mathrm{e}_p = 0$, $p = 1, \dots, m$, are the Euler equations).

PROPOSITION 3. *For any Lagrangian $\lambda$ and any splitting $S$, if an $r$-holonomic section $u : X \to J^{2r-1}Y$ satisfies $u^*(\zeta \lrcorner \mathrm{d}\theta_S) = 0$ for every $\pi_{2r-1}$-vertical vector field $\zeta$ on $J^{2r-1}Y$, then $s = \pi^0_{2r-1}u$ is a critical section of $\lambda$.*

*Proof.* We have $\theta_S = \lambda + \sum_{|\alpha| \leqslant r-1} b^{\alpha i}_p \omega^p_\alpha \wedge \omega_i$, so that

(32) $$\zeta \lrcorner \mathrm{d}\theta_S = \zeta \lrcorner \mathrm{d}\lambda + \sum_{|\alpha| \leqslant r-1} [(\zeta \lrcorner \mathrm{d}b^{\alpha i}_p)\omega^p_\alpha \wedge \omega_i - \zeta^p_\alpha \mathrm{d}b^{\alpha i}_p \wedge \omega_i - b^{\alpha i}_p \zeta^p_{\alpha+i}\omega].$$

Since $u$ is $r$-holonomic, the second term on the right-hand side vanishes on $u$. Hence $u^*(\zeta \lrcorner \mathrm{d}\theta_S) = 0$ is equivalent to

(33) $$((j^r s)^* L^{j_1 \dots j_k}_p)\omega = u^*(\mathrm{d}B^{j_1 \dots j_k i}_p \wedge \omega_i) + (u^* B^{(j_1 \dots j_k)}_p)\,\omega,$$

$k = 0, \dots, r-1$. For $k = r$, $B^{j_1 \dots j_r}_p = L^{j_1 \dots j_r}_p$ are some functions on $J^rY$, so that

(34) $$u^* B^{j_1 \dots j_{k+1}}_p = (j^{2r-k-1}s)^* B^{j_1 \dots j_{k+1}}_p$$

for $k + 1 = r$. Assume by induction that (34) holds for some $k+1$ and we prove that (34) holds for $k$ as well. Since $\mathrm{d}((j^{2r-k-1}s)^* B^{j_1 \dots j_{k+1}}_p) = (j^{2r-1}s)^* \cdot DB^{j_1 \dots j_{k+1}}_p$ by the definition of the formal exterior differential, we have $u^*(\mathrm{d}B^{j_1 \dots j_k i}_p \wedge \omega_i) = \mathrm{d}((j^{2r-k-1}s)^* B^{j_1 \dots j_k i}_p) \wedge \omega_i = (j^{2r-k}s)^* DB^{j_1 \dots j_k i}_p \wedge \omega_i$. But

$B$'s satisfy (7), so that (33) implies $u^*B_p^{(j_1 \dots j_k)} = (j^{2r-k}s)^* B_p^{(j_1 \dots j_k)}$. By (19), $B_p^{j_1 \dots j_k}$ differs from $B_p^{(j_1 \dots j_k)}$ by some linear combinations with coefficients defined on $X$ of $B$'s with more superscripts and some their formal derivatives. But all these quantities are defined on $J^{2r-k-1}Y$ and their values on $u$ coincide with their values on $j^{2r-k-1}s$ by the induction hypothesis and by the definition of the formal derivative. Hence (34) holds for $k$ superscipts as well. Thus, by induction, (34) is true for all $B$'s. By (33) and (34), $u^*(\zeta \rfloor d\theta_S) = 0$ is reduced to

$$(j^r s)^* L_p = (j^{2r}s)^* D_i B_p^i, \tag{35}$$
$$(j^r s)^* L_p^{j_1 \dots j_k} = (j^{2r-k}s)^* D_i B_p^{j_1 \dots j_k i} + (j^{2r-k}s)^* B_p^{(j_1 \dots j_k)}$$

$k = 1, \dots, r-1$. Using $B_p^{j_1 \dots j_r} = L_p^{j_1 \dots j_r}$ and the backward elimination, we deduce $(j^{2r}s)^* e_p = 0$. Q.E.D.

REMARK 2. Proposition 5 of [6] implies that every critical section $s$ satisfies $(j^{2r-1}s)^*(\zeta \rfloor d\theta_S) = 0$ for any splitting $S$ and any $\pi_{2r-1}$-vertical vector field $\zeta$ on $J^{2r-1}Y$.

A geometrical version of the higher order Hamilton formalism can be now formulated as follows.

PROPOSITION 4. *Let $\lambda$ be a regular $r$-th order Lagrangian on $Y \to X$ and $S : T^*X \to T^{r-1*}X$ be any splitting. If a section $u : X \to J^{2r-1}Y$ satisfies $u^*(\zeta \rfloor d\theta_S) = 0$, then $s = \pi^0_{2r-1} u$ is a critical section of $\lambda$.*

*Proof.* Since $\lambda$ is regular, $u$ is $r$-holonomic by Proposition 2. Then $s$ is critical by Proposition 3. Q.E.D.

## REFERENCES

[1] M. FERRARIS, *Fibred connections and global Poincaré-Cartan forms in higher order calculus of variations*, to appear in Proceedings of the conference on differential geometry and its applications, Nové Město na Moravě, September 5 - 9, 1983.

[2] M. FRANCAVIGLIA, D. KRUPKA, *The Hamilton formalism in higher order variational problems*, Ann. Inst. Henri Poincaré, XXXVII (1982), 295 - 315.

[3] P.L. GARCÍA, *The Poincaré-Cartan invariant in the calculus of variations*, Instituto Nazionale di Alta Matematica, Symposia Mathematica XIV, Roma 1974, 219 - 246.

[4] P.L. GARCÍA, J. MUÑOZ, *On the geometrical structure of higher order variational calculus*, Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics, Torino, June 7 - 11, 1982, Tecnoprint, Bologna 1983, 127 - 147.

[5] H. GOLDSCHMIDT, S. STERNBERG, *The Hamilton formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble), 23 (1973) 203 - 267.

[6] M. HORÁK, I. KOLÁŘ, *On the higher order Poincaré-Cartan forms*, Czechoslovak Math. J., 33 (108) (1983), 467 - 475.
[7] M. HORÁK, *On the geometry of the higher order Lagrange and Hamilton formalisms in fibered manifolds (in Czech)*, PhD thesis, Brno 1982.
[8] I. KOLÁŘ, *On the Hamilton formalism in fibered manifolds*, Scripta Fac. Sci. Nat. UJEP Brunensis, Physica 5, 1975, 249 - 254.
[9] I. KOLÁŘ, *Some geometric aspects of the higher order variational calculus*, to appear in Proceedings of the conference on differential geometry and its applications, Nové Město na Moravě, September 5 - 9, 1983.
[10] D. KRUPKA, *Lepagean forms in higher order variational theory*, Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics, Torino, June 7 - 11, 1982, Tecnoprint, Bologna 1983, 197 - 238.
[11] M. MODUGNO, G. STEFANI, *Some results on jet spaces*, Quaderni dell'Istituto di Matematica dell'Università di Lecce, Q. 17, 1978.
[12] W.F. SHADWICK, *The Hamiltonian formulation of regular r-th order Lagrangian field theories*, Letters in Mathematical Physics **6** (1982), 409 - 416.
[13] A. TRAUTMAN, *Invariance of Lagrangian systems, General Relativity*, Papers in honour J.L. Synge, Clarendon Press, Oxford 1972, 85 - 99.

*Manuscript received: May 24, 1984.*